LINEAR POLARIZATION CONSTANT OF $\mathbb{R}^n$

MÁTÉ MATOLCSI

Abstract. The present work contributes to the determination of the n-th linear polarization constant $c_n(H)$ of an n-dimensional real Hilbert space H. We provide some new lower bounds on the value of $\sup_{\|y\|=1} |\langle x_1, y\rangle \cdots \langle x_n, y\rangle|$, where $x_1, \dots, x_n$ are unit vectors in H. In particular, the results improve an earlier estimate of Marcus. However, the intriguing conjecture $c_n(H) = n^{n/2}$ remains open.

2000 Mathematics Subject Classification. Primary 46G25; Secondary 52A40, 46B07.

Keywords and phrases. *Polynomials over normed spaces, linear polarization constants, Gram matrices*

1. Introduction

In this note we aim to make a contribution to estimating the n-th linear polarization constant $c_n(H)$ of an n-dimensional real Hilbert space H. We begin with introducing some (more general) standard terminology and giving a short account of some related results.

Let X denote a Banach space over the real or complex field $\mathbb{K}$. A function $P: X \to \mathbb{K}$ is a continuous *n-homogeneous polynomial* if there exists a continuous n-linear form $L: X^n \to \mathbb{K}$ such that $P(x) = L(x, \dots, x)$ for all $x \in X$. We define

$$\|P\| := \sup\{|P(x)| : x \in B\}$$

where B denotes the unit ball of X. Considerable attention has been devoted to polynomials of the form $P(x) = f_1(x) f_2(x) \dots f_n(x)$, where $f_1, f_2, \dots, f_n$ are bounded linear functionals on X. For any *complex* Banach space X Benítez, Sarantopoulos and Tonge [3] have obtained

$$\|f_1\| \, \|f_2\| \cdots \|f_n\| \leq n^n \|f_1 f_2 \dots f_n\|,$$

and the constant n^n is best possible. For *real* Banach spaces, Ball's solution [6] of the famous plank problem of Tarski gives the same general result. For specific spaces, however, the general constant n^n can be lowered.

Definition 1.1. *(Benítez, Sarantopoulos, Tonge [3]) The n-th linear polarization constant of a Banach space X is defined by*

$$\begin{aligned} c_n(X) &:= \inf\{M : \|f_1\| \cdots \|f_n\| \leq M\|f_1 \cdots f_n\| \ (\forall f_1, \ldots, f_n \in X^*)\} \\ &= 1/ \inf_{f_1,\ldots,f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x) \cdots f_n(x)|. \end{aligned}$$

The linear polarization constant of X is defined by

$$c(X) := \lim_{n\to\infty} c_n(X)^{\frac{1}{n}} \quad . \tag{1}$$

Let us recall that the above definition of $c(X)$ is justified since Révész and Sarantopoulos [2] showed that the limit (1) does exist. Moreover, they also showed (both in the real and complex cases) that $c(X) = \infty$ if and only if $\dim X = \infty$.

Note that it is easy to see that for any Banach space X we have

$$c_n(X) = \sup\{c_n(Y) : Y \text{is a closed subspace of} \, X, \, \dim Y = n\} \ . \tag{2}$$

In particular, for a real or complex Hilbert space H of dimension at least n, we always have $c_n(H) = c_n(\mathbb{K}^n)$.

Benítez, Sarantopoulos and Tonge [3] proved that for isomorphic Banach spaces X and Y we have $c_n(X) \leq d^n(X,Y)c_n(Y)$, where $d(X,Y)$ denotes the Banach-Mazur distance of X and Y. Note, that for any n-dimensional space X a result of John [8] states that $d(X, \mathbb{K}^n) \leq \sqrt{n}$ (where $\mathbb{K}^n$ denotes the n-dimensional Hilbert space). The combination of these results mean that the determination of $c_n(\mathbb{K}^n)$ gives information on the linear polarization constants of other spaces, too.

Here we are going to focus our attention to Hilbert spaces. Pappas and Révész [4] showed that $c(\mathbb{K}^n) = e^{-L(n,\mathbb{K})}$, where

$$L(n, \mathbb{K}) := \int_S \log|\langle x, e\rangle| d\sigma(x);$$

here S and σ denote the unit sphere and the normalized surface measure, respectively, and $e \in S$ is an arbitrary unit vector. This result gives information on the asymptotic behaviour of $c_m(\mathbb{K}^n)$ as $m \to \infty$. However, the exact values of $c_m(\mathbb{K}^n)$ seem very hard to determine. Anagnostopoulos and Révész [11] found explicit relations between $c_n(\mathbb{R}^2)$, $c_n(\mathbb{C}^2)$ and the Chebyshev constants of S^1 and S^2, respectively. These relations, however, do not seem to carry over to higher dimensional spaces. Note that the n-th Chebyshev constant of S^1 is well-known, but the exact determination of the n-th Chebyshev constant of S^2 seems hopeless; see [11] and [10].

A remarkable result of Arias-de-Reyna [7] states that $c_n(\mathbb{C}^n) = n^{n/2}$. Ball's recent solution [5] of the complex plank problem also implies the same result.

The value of $c_n(\mathbb{R}^n)$ seems even harder to find. The determination of $c_n(\mathbb{R}^n)$, by the definition and the Riesz representation theorem, boils down to determining

$$I := \inf_{x_1,\ldots,x_n\in S} \sup_{\|y\|=1} |\langle x_1, y\rangle \cdots \langle x_n, y\rangle|$$

The estimate $I \le n^{-\frac{n}{2}}$ follows by considering an orthonormal system.

The result of Arias-de-Reyna can be used to derive the following estimates [2]:

$$n^{\frac{n}{2}} \le c_n(\mathbb{R}^n) \le 2^{\frac{n}{2}-1} n^{\frac{n}{2}}.$$

A natural, intriguing conjecture, see [3], [2] is the following.

Conjecture. $c_n(\mathbb{R}^n) = n^{n/2}$.

Marcus (communicated in [9], and elaborated later in [2]) gives the following estimate: If $x_1, x_2, \ldots, x_n$ are unit vectors in $\mathbb{R}^n$ then there exists a unit vector y such that

$$|\langle x_1, y\rangle \cdots \langle x_n, y\rangle| \ge (\lambda_1/n)^{n/2}, \tag{3}$$

where λ_1 denotes the smallest eigenvalue of the Gram matrix $XX^* = [\langle x_i, x_j\rangle]$. Marcus also expressed the opinion that lower bounds on $\sup_{\|y\|=1} |\langle x_1, y\rangle \cdots \langle x_n, y\rangle|$ should involve the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Gram matrix $XX^* = [\langle x_i, x_j\rangle]$, i.e. we should look for estimates of the form $\sup_{\|y\|=1} |\langle x_1, y\rangle \cdots \langle x_n, y\rangle| \ge f(\lambda_1, \ldots, \lambda_n) n^{-n/2}$. Note that $\sum_j \lambda_j = \operatorname{Tr} XX^* = n$. Therefore the above Conjecture can be equivalently formulated as

$$\sup_{\|y\|=1} |\langle x_1, y\rangle \cdots \langle x_n, y\rangle| \ge 1 \cdot n^{-n/2} = \left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)^{n/2} n^{-n/2}.$$

In the next section we show that if we replace the arithmetic mean by the harmonic mean of the numbers $\lambda_1, \ldots, \lambda_n$, then the corresponding lower estimate does hold. This gives an improvement of Marcus' result but the Conjecture still remains open.

2. Lower bounds

In this section we are going to present three different lower bounds on the value of $\sup_{\|y\|=1} |\langle x_1, y\rangle \cdots \langle x_n, y\rangle|$. The first result relies on an averaging argument, while the next two uses the following lemma of Bang [1]:

Lemma 2.1. *Let $H = (h_{jk})$ be an $n \times n$ Gram matrix and $r_1, \dots r_n$ be a sequence of positive numbers. Then there are signs $\varepsilon_1, \dots \varepsilon_n$ for which*

$$\varepsilon_j r_j \sum_{k=1}^{n} h_{jk} r_k \varepsilon_k \geq r_j^2 \tag{4}$$

for every j.

Assume now, that we are given n unit vectors $x_1, \dots, x_n$ in $\mathbb{R}^n$. We are looking for a unit vector y such that the product $| \langle x_1, y \rangle \cdots \langle x_n, y \rangle |$ is 'as large as possible'. Let X denote the $n \times n$ matrix whose j-th row is x_j. With this notation, our aim is to maximize the expression $\prod_{j=1}^{n} |(Xy)_j|$. As a first observation we 'symmetrize' the matrix X. Take the polar decomposition $X^* = V(XX^*)^{1/2}$ of X^*. The partial isometry V maps $(\ker X^*)^{\perp}$ to $\operatorname{Im} X^*$, and can be extended (not uniquely, in general) to a unitary operator U. Taking adjoints we get $X = (XX^*)^{1/2} U^*$. The unitary operator U^* maps the unit sphere onto itself, therefore

$$\max_{\|y\|=1} \prod_{j=1}^{n} |(Xy)_j| = \max_{\|y\|=1} \prod_{j=1}^{n} |((XX^*)^{1/2} y)_j|.$$

Therefore, in the forthcoming arguments, we are going to consider the positive self-adjoint matrix $(XX^*)^{1/2}$ instead of the original matrix X. Note, also, that the polar decomposititon implies that the rows (and , by symmetry, the columns) of the matrix $(XX^*)^{1/2}$ are also unit vectors

Theorem 2.2. *Assume that the given unit vectors $x_1, \dots, x_n$ are linearly independent. Let $V_1, \dots, V_n$ denote the lengths of the column-vectors of the matrix $(XX^*)^{-1/2}$. Then*

$$\max_{\|y\|=1} \prod_{j=1}^{n} |(Xy)_j| \geq \frac{1}{V_1 \cdots V_n} \cdot n^{-\frac{n}{2}} \tag{5}$$

Proof. Note that the inverse matrix $(XX^*)^{-1/2}$ exists by the assumption of linear independency. Denote the entries of $(XX^*)^{-1/2}$ by (v_{jk}).

Take any vector $c := (c_1, \dots, c_n)^T$ with $c_j \geq 0$ $(j = 1, \dots, n)$ and consider all possible 2^n arrangements of signs of the entries in c: i.e., consider $c(\varepsilon) := (\varepsilon_1 c_1, \dots, \varepsilon_n c_n)^T$ for all $\varepsilon \in \{\pm 1\}^n$. We evaluate the sum $\sum_{\varepsilon} \|(XX^*)^{-1/2} c(\varepsilon)\|^2$. It is easy to see that the double products $2c_j v_{ij} c_k v_{ik}$ all cancel out because they appear an equal number of times with positive and negative signs. Therefore,

$$\sum_{\varepsilon} \|(XX^*)^{-1/2} c(\varepsilon)\|^2 = 2^n \cdot \sum_{jk} c_k^2 v_{jk}^2 = 2^n \cdot \sum_{k} c_k^2 V_k^2 \tag{6}$$

Hence, there exists a signed vector $c(\varepsilon) = (\varepsilon_1 c_1, \dots, \varepsilon_n c_n)^T$ such that $\|(XX^*)^{-1/2}c(\varepsilon)\| \le (\sum_k c_k^2 V_k^2)^{1/2}$. Take

$$y := \frac{(XX^*)^{-1/2}c(\varepsilon)}{\|(XX^*)^{-1/2}c(\varepsilon)\|}.$$

Then

$$\prod_{j=1}^{n} |((XX^*)^{1/2}y)_j| \ge \prod_j c_j \cdot (\sum_k c_k^2 V_k^2)^{-n/2}. \tag{7}$$

After introducing the new variables $b_1 := c_1V_1, \dots b_n := c_nV_n$, the right hand side becomes $\prod_j b_j \cdot (\sum_k b_k^2)^{-n/2} \cdot \prod_j V_j^{-1}$. By the inequality of the quadratic and geometric means this expression is maximal if and only if $b_1 = b_2 = \cdots = b_n$, which is achieved by the choice $c_j := V_j^{-1}$. Substituting $c_j := V_j^{-1}$ in (7) we arrive at the required inequality (5). □

We remark that $V_j \ge 1$ for every j, and $V_j = 1$ for all j if only $x_1, \dots, x_n$ forms an orthonormal system, hence in general we cannot prove the Conjecture by this argument. It is easy to see, however, that our result is stronger than the estimate (3) of Marcus.

Corollary 2.3. *Assume that the given unit vectors $x_1, \dots, x_n$ are linearly independent. Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the Gram matrix XX^*. Then*

$$\max_{\|y\|=1} \prod_{j=1}^{n} |(Xy)_j| \ge \left(\frac{n}{\lambda_1^{-1} + \cdots + \lambda_n^{-1}}\right)^{n/2} \cdot n^{n/2} \tag{8}$$

Proof. Observe that $\sum_k V_k^2 = \sum_{jk} v_{jk}^2 = \mathrm{Tr}\,(XX^*)^{-1} = \lambda_1^{-1} + \cdots + \lambda_n^{-1}$. Therefore

$$\frac{1}{V_1 \cdots V_n} \ge \left(\frac{n}{\sum_k V_k^2}\right)^{n/2} = \left(\frac{n}{\lambda_1^{-1} + \cdots + \lambda_n^{-1}}\right)^{n/2}.$$

□

The next two estimates are in the same spirit but the proofs rely on Bang's lemma instead of the averaging technique of Theorem 2.2.

Theorem 2.4. *Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the Gram matrix XX^* in increasing order. Then*

$$\max_{\|y\|=1} \prod_{j=1}^{n} |(Xy)_j| \ge \left(\frac{1}{\lambda_n}\right)^{n/2} \cdot n^{n/2} \tag{9}$$

Proof. Let $\varepsilon := (\varepsilon_1, \dots \varepsilon_n)$ be any sequence of signs, and let E denote the diagonal matrix with the numbers $(\varepsilon_j)_{j=1}^n$ in the diagonal. Bang's lemma (applied to the numbers $r_1 = r_2 = \dots = r_n = 1$) implies that there exists a sequence of signs ε such that $EXX^*E(1, 1, \dots, 1)^T \geq (1, 1, \dots, 1)^T$. Take

$$y := \frac{(XX^*)^{1/2}\varepsilon}{\|(XX^*)^{1/2}\varepsilon\|}.$$

Then

$$\prod_j |((XX^*)^{1/2}y)_j| \geq \left(\frac{1}{\|(XX^*)^{1/2}\varepsilon\|}\right)^n \geq (n\lambda_n)^{-n/2} \tag{10}$$

where the last estimate follows from $\|\varepsilon\| = n^{1/2}$ and $\|(XX^*)^{1/2}\| = \lambda_n^{1/2}$. □

We remark that the choice of y above actually satisfies the inequality $\prod_j |((XX^*)^{1/2}y)_j| \geq n^{-n/2}$ (proving also the Conjecture) for $1 \leq n \leq 5$, as shown in [4]. It is not clear, however, whether this particular choice works also for larger values of n. Note that $\lambda_n \geq 1$, and Theorem 2.4 yields the conjectured estimate only in case of $\lambda_n = 1$, that is, only in the case of an orthonormal system.

Theorem 2.5. *Let $a_1, \dots, a_n$ denote the diagonal entries of the matrix $(XX^*)^{1/2}$. Then*

$$\max_{\|y\|=1} \prod_{j=1}^n |(Xy)_j| \geq a_1 \cdots a_n \cdot n^{-n/2}\,. \tag{11}$$

Proof. Let A denote the $n \times n$ diagonal matrix with $(a_j)_{j=1}^n$ in the diagonal. The matrix $B := A^{-1/2}(XX^*)^{1/2}A^{-1/2}$ is positive, self-adjoint and has 1's in the main diagonal, i.e. it is a Gram matrix. Apply Bang's lemma to B with numbers $a_1^{1/2}, \dots, a_n^{1/2}$. We conclude that there exists a choice of signs $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n)$ (with the corresponding diagonal matrix E) such that $EA^{1/2}BA^{1/2}E(1, 1, \dots, 1)^T \geq (a_1, a_2, \dots, a_n)^T$ (coordinatewise). This means that the choice $y := n^{-1/2}(\varepsilon_1, \dots, \varepsilon_n)^T$ gives $\prod_j |((XX^*)^{1/2}y)_j| \geq a_1 \cdots a_n \cdot n^{-\frac{n}{2}}$. □

Note that $a_j \geq \lambda_1^{1/2}$ for every j, therefore the result above is also an improvement on Marcus' estimate.

Let us conclude with the following remarks.

The three different estimates presented in Theorems 2.2, 2.4 and 2.5 do not seem to be comparable; at least we were not able to show that any of them would imply another. An advantage of the proofs applied is that in all three theorems we were able to pinpoint our choice of the vector y. It is clear, however, that none of these estimates settles the Conjecture.

References

[1] T. Bang: A solution of the plank problem, *Proc. Amer. Math. Soc.* **2**(1951), 990-993.

[2] Sz. Révész, Y. Sarantopoulos: Plank problems, polarization and Chebyshev constants, *Korean J. Math.*, to appear.

[3] C. Benitez, Y. Sarantopoulos, A.M Tonge: Lower bounds for norms of products of polynomials, *Math. Proc. Cambridge Philos. Soc.* **124**(1998), 395-408.

[4] A. Pappas, Sz Révész: Linear polarization constants of Hilbert spaces, *manuscript* (2002).

[5] K. M. Ball: The complex plank problem, *Bull. London Math. Soc.* **33**(2001), 433-442.

[6] K.M. Ball: The plank problem for symmetric bodies, *Invent. Math.* **104**(1991), 535-543.

[7] J. Arias-de-Reyna: Gaussian variables, polynomials and permanents, *Linear Algebra Appl.* **285**(1998), 107-114.

[8] F. John: Extremum problems with inequalities as subsidiary conditions, *Courant Aniversary Volume, Interscience, New York* (1948), 187-204.

[9] M. Marcus: Letter to Y. Sarantopoulos, (1996).

[10] G. Wagner: On the product of distances to a point set on a sphere. *J. Australian Math. Soc.,(Ser. A)* **47** (1989), 466-482.

[11] V. Anagnostopoulos, Sz. Révész: Polarization constants for products of linear functionals over $\mathbb{R}^2$ and $\mathbb{C}^2$ and Chebyshev constants of the unit sphere. *manuscript*, (2001).

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences POB 127 H-1364 Budapest, Hungary Tel: (+361) 483-8302, Fax: (+361) 483-8333

E-mail address: `matomate@renyi.hu`